Lecture 1: Euler's Disk, or the precession of a spolling disk

A heuristic solution for the precession of Euler's Disk reconstructed from notes taken during an informal presentation with co-workers (~1989): We proceed as follows to find (or better understand) something called the precession (Ω) of Euler's Disk. No pictures here, but since were not using the term "vector" during the discussion, we shouldn't need one – right?

For what follows below, we must first be convinced that as the disk falls or spolls (to spin and roll) into a "very" shallow angle, that energy is lost or dissipated – even though the disk sounds like its speeding up. Some ponder and ask, "If the disk *speeds up* isn't the disk increasing its energy or maybe converting PE into KE?" Here, the teacher can simply recite some of the following lines of reasoning derived from several classic paradoxes, like the old, "ping-pong ball paradox." The classic case where a ping-pong ball dropped from a few feet over a hard surface appears to move faster and faster as it bounces over and over again, just before it abruptly stops bouncing (the teacher explains that the maximum velocity (energy) for each bounce actually decreases as the path grows ever shorter ~ Zeno's paradox). Or maybe, more importantly for the case at hand, one can envision opening or closing a very long, (imaginary) pair of scissors, very quickly. Here the teacher simply "points out" that the moving point of contact (mpc) between the blades, does in fact, move-off at a very high speed, but the mpc is only a geometric construction possessing zero energy (not a real material object). The mpc for an Euler's disk works in a similar way.

Our proposal:

For the one, very troubled student who refuses to accept that energy is lost *during* the fall, we start with the equation for the total energy of the disk (or, E= KE+PE) and explain this is a finite quantity at the start of motion. Next, we assume initially that: the edge of the disk is rolling (with minimal vibration) against a flat glass base in a tight circle (with radius ~ = disk radius), the disk face makes a small angle (α) with the glass surface, and is observed to fall slowly to the base (i.e., the disk center of mass, CoM, moves ~ straight down - slowly). The student suggests that PE is simply converted into KE as the disk falls causing an increase in both the angular velocity and/or CoM velocities near the end of motion. He writes down something taken from his Feynmnn book like; $E \sim \frac{1}{2} I\omega^2 + mgz + KE$ (CoM) where ω, is the (absolute) angular velocity of the disk and z, is the height of the CoM from the glass base. One can "point out" (from observation) that the velocity of the disk (CoM) is small and although it might increase* as the disk falls to the base, it doesn't seem to be responsible (in terms of energy) for the increasing *oscillation* - maybe it is the rotational energy that's increasing (?). In the end, the argument that the disk loses most of its energy as it falls is (possibly) accepted by our reluctant student when we explain the mechanism required for increasing the "rotational energy" of our disk. We mention how it would take a *special* torque to do useful work and change the *value* of angular velocity, ω. The only torque available is derived by gravity (normal force created by the gravitational field) which for the case at hand, does not produce useful work (vide infra).

Physics/dynamics:

Moving on, we now assume that our disk is spolling at a small, *constant* angle α for a very long time (the CoM is fixed and no longer falling – it slips into a final position – friction is negligible). We can call this motion - harmonic - if we like. For our special case here we must first try to convince ourselves that the instantaneous motion of the disk can be completely described by a simple rotation about an instantaneous axis of rotation (IAR pronounced “ear”). This “special” axis lies along the line between the instantaneous point of contact and the fixed CoM (this “ear” can be visualized by rotating the disk on edge, at a shallow angle). Now, the angular momentum (L) of a disk about this special axis (IAR) is simply, $I\omega$. Where “I” (given by the formula -> ma^2/4) represents the moment of inertial about the IAR and “ω” is the angular velocity about this same (principal) axis.

Now, since the disk spolls around a tight circle, the IAR moves about a tight circle along with L. So, as the disk spolls, L continuously changes its position (direction). This change in L with time (or dL/dt) stems solely from an externally applied torque** (T=dL/dt). The torque acting on our disk (about the stationary CoM) is derived by taking the product of the normal force (~mg, derived from gravity) pushing up on the edge of the disk (from the glass base) with an appropriate lever distance (~a, in our case) or T = ~mga (we might, for demonstration purposes, say gravity is tending to torque the disk about the point of contact…but, since this is not a simple “gyro/top” a more suitable origin is the CoM). Either way, since torque acts as a “right-hand-lever,” the (instantaneous) *change* in angular momentum (dL/dt) of the disk is perpendicular to both the applied normal force and the IAR containing L. Since we know that L moves about a circle, we have a good sense in how and in what direction L changes (instantaneously) – even if the disk is moving rather quickly. In the end, our torque produces no useful work since it does not act *about* the IAR, but, it does continuously *force* the disk to precess about the glass table – a dramatic difference as compared with a stationary disk falling / rotating down on its side! Now, one final point, even though the angular velocity of the instantaneous point of contact moves about a circle along with the IAR and L, the physical edge of the disk does not slide around glass base since the disk is (assumed for discussion) rolling (you can hear something like rolling) and by definition the “instantaneous velocity of the edge of the disk in contact with the base is zero.”

Pushing our limits:

Recall from basic geometry that a “small arc” or small distance (something like our instantaneous *change* in L) can be expressed as the product of the radius of a circle (something like L) and an angular quantity (something like Ω) about a vertical axis. So, since we know the disk spolls in a circle, (changing the direction of the angular momentum L continuously without loss of energy) we might conjecture that its possible to express this (instantaneous) *change* in L (dL/dt = T = mga) as a product of L with a proper angular term (precession) “Ω.” All this just means is that its ok for the teacher to set dL/dt equal to ΩL. Once this is done, we can immediately solve for the precession term: Ω(radians/sec) = (dL/dt) / L = 4mga/ωma^2 = 4g/aω. Our precession term represents both the (angular) rate at which the point of contact moves about the glass base and a scale factor representing how much torque is applied to the disk CoM (i.e., the disk and angular momentum both precess about a vertical axis at a rate Ω).

Now, if "ω" goes to zero, (if we let the disk spoll down a bit!) we can see at once that Ω, would diverge as $1/\omega$. If "ω" were to increase or stay constant (per our troubled students demand) then our precession would go to zero, or stay fixed. Now again, the only way to pump energy into the disk via the absolute angular velocity (ω) is to apply a special torque – but alas, there is only one torque about our CoM disk system (the normal force, mg) and we have shown that it pushes the IAR around a circle and can not push the disks mass about the IAR. So I think we have proved to our troubled student that a real (non-harmonic) disk really does loose energy as it spolls down in practice and that the precession (Ω) goes to infinity (as $1/\omega$) – even though some energy might be converted into CoM velocity*. We simply can not find a way to convert our PE into *rotational* KE; even his suggestion that the rotational KE might be constant is suspect - since we know that the rotational quantity of interest apparently increases as the disk falls. A rigorous proof showing how Ω and ω go to infinity/zero as (root) α requires several extended tricks.

We really should be quite pleased with our simple derivation for our angular term or precession (Ω) since along the way we faced several apparently interrelated paradoxes. For example, we might have started thinking down these lines – if the instantaneous velocity of the point of contact of the disk with the base and the CoM are both zero, doesn't that mean all material points along that line are not moving? Yet, the precession about a vertical axis heads to infinity – so what is part of the disk is moving around? Or maybe, why do spots on the surface of the disk appear to rotate more slowly as the disk falls to smaller angles (?) – I thought we only have rotations about the ear and vertical axis? Or, can the disk roll without friction (a favorite)? At this point, I hope the first and second examples are partially resolved, since we focused most of our attention on the motion of the "ear" and the "attached" rolling disk.

The troubled student (and teacher) might also request more clarification concerning the disk's precession and the overall state of motion of the disk (pseudorotation or spolling). At this point it might be nice to demonstrate how our disk can be put into any instantaneous (vide supra) spolling position by considering two separate and important rotations (even though we might use Euler's Theorem and say "one"). The first, a simple rotation of the disk on edge (without *spin*) about an axis perpendicular to the table (to demo: hold the disk at the proper spolling angle and rotate it about an axis perpendicular to the table). Here, only the edge of the disk slides around the table as the disk is rotated. The second rotation involves keeping the spolling angle fixed and spinning the disk on edge. Here, many points on the edge of the disk touch the table during the rotation. The combination of these two motions and some knowledge of the position of the CoM can produce the effect of our rolling/spinning (spolling) disk for the special case at hand and more generally as well.

Finally, placing a sticker on the face/edge of the disk and watching it "slow down" over time is a neat example of a "projection-difference" of angular velocities as compared with more common examples of "addition" of angular velocities (1970's SAT tests for example – rolling a quarter about a quarter). One can show that the angular speed of this apparent rotation is just $\Omega(1-\cos(\alpha))$ and this sticker rotates forwards or backwards (oddly enough) given different spolling conditions (rolling radius >< disk radius).

In fact, more generally, one can summon two (or more) reference frames (e.g., a fixed lab frame, a frame attached to the disk with or without spin) to describe the motion of Euler's Disk more completely, but this unfortunately requires projecting rotational components onto moving axes - using the unthinkable, "vectors."

JBendik (99/00/05/07)

P.S. for now, our brief chat concerning the kinematics and dynamics of an Euler's Disk is finished. Remember, we still have two more discussions: one concerning the acoustics of the toy and another for something called "optical tessellation." For those interested in learning more about disk rotation I have a handout with a set of coin problems titled, "Travels with Euler."

* $|\ddot{\alpha} a| \leq g$ increases as the disk falls, see "Intro-Space-Dynamics," W. Thomsom, p. 154.

** see "Fowles," p. 201-203.